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An Inversion Model for q-Identities

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1. INTRODUCTION

The purpose of this paper is to present a combinatorial treatment of a number of the classical (number) partition identities which involve finite products. Our treatment has been directed by two aims. The first is that the treatment of the q-identities considered here should specialize combinatorially to the usual proofs of the corresponding binomial identities ($q = 1$). The second is that the Gaussian coefficient $\binom{n}{k}_q$ should have a natural combinatorial interpretation.

The model we examine is different from that of Goldman and Rota [5], who considered the enumeration of certain classes of linear transformations on vector spaces over $GF(q)$, where q is a prime power. Instead, we consider the enumeration of particular classes of permutations (in fact cup- and cap-permutations) with respect to the number of inversions they contain. In this theory, q is an indeterminate, and has the interpretation that $[q^k]f(q)$, the coefficient of q^k in an appropriate generating function, gives the number of permutations, in some set, which have exactly k inversions.

In the model used by Andrews [1], q is also an indeterminate marking inversions. However, the interpretation given to $\binom{n}{k}_q$ in [1] does not allow a convenient treatment of the finite product $Q_n(z, x) = (x+z)(x+qz)\cdots(x+q^{n-1}z)$, as our interpretation does.

There has been a great deal of recent interest in the enumeration of permutations with respect to number of inversions and other characteristics, and in the q-analogues of combinatorial numbers in general. (See, for example, Garsia and Gessel [2], Garsia and Remmel [3], Gessel [4], Jackson and Goulden [6], Stanley [7].) However, the combinatorial interpretation of $Q_n(z, x)$ and its use in deriving the q-analogue of the binomial theorem (Goldman and Rota [5]), a special case of our main result (Theorem 4.2), seem to have been overlooked.

2. NOTATION

The following notational conventions are used throughout this paper. If $k \geqslant 1$ is an integer then $k!_q$ denotes the expression $\prod_{i=1}^{k}(1-q^i)/(1-q)$, $0!_q = 1$ and $\binom{n}{k}_q$ denotes the *Gaussian coefficient* $n!_q\{(n-k)!_q k!_q\}^{-1}$. The multinomial generalization of this is denoted by $\begin{bmatrix} n \\ \underset{\sim}{i} \end{bmatrix}_q$, where $\underset{\sim}{i} = (i_1, \ldots, i_k)$ and $i_1 + \cdots + i_k = n$, and is defined to be $n!_q(i_1!_q \cdots i_k!_q)^{-1}$.

If $\alpha = \{\alpha_1, \ldots, \alpha_r\} \subseteq \mathcal{N}_n$, where $\mathcal{N}_n = \{1, \ldots, n\}$, then $(\alpha)_<$ denotes the *increasing permutation* $\alpha_{i_1} \cdots \alpha_{i_r}$ on α, where $\alpha_{i_1} < \cdots < \alpha_{i_r}$ and $\{i_1, \ldots, i_r\} = \mathcal{N}_r$. The permutation $(\alpha)_>$, similarly defined is a *decreasing permutation*.

Let $\underset{\sim}{i} = (i_1, \ldots, i_r)$ and $\Pi = (\pi_1, \ldots, \pi_r)$ be an ordered set partition of $\mathcal{N}_n$ with $|\pi_j| = i_j$ for $j = 1, \ldots, r$. Then Π is said to have *type* $\underset{\sim}{i}$. If $\rho = \rho_1 \cdots \rho_r$ is a permutation on $\mathcal{N}_n$ such that ρ_j is a permutation on π_j for $j = 1, \ldots, r$, then Π is the partition of type $\underset{\sim}{i}$ associated with ρ.

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If (α, β) is an ordered bipartition of $\mathcal{N}_n$ of type (i, j), then $(\alpha)_>(\beta)_<$ (respectively $(\alpha)_<(\beta)_>$) is called a *cup-* (respectively *cap*)-*permutation* of *shape* (i, j).

3. The Gaussian Coefficient

We present, first, some basic properties of inversions in permutations.

Proposition 3.1.

(1) *Let* $\sigma = \sigma_1 \cdots \sigma_k$ *where* $\{\sigma_1, \ldots, \sigma_k\} \subseteq \mathcal{N}_n$. *Let* $I(\sigma)$ *denote the number of inversions (a pair* (σ_i, σ_j) *with* $\sigma_i > \sigma_j$ *and* $i < j$*) in* σ. *If* $\phi : \mathcal{N}_n \to \mathcal{N}_n$ *is order preserving then* $I(\phi(\sigma_1) \cdots \phi(\sigma_k)) = I(\sigma_1 \cdots \sigma_k)$.

(2) *Let* $\sigma = \sigma_1 \cdots \sigma_n$ *be a permutation on* $\mathcal{N}_n$ *such that* $\sigma_j = n$. *If* $\hat{\sigma}$ *is the permutation on* $\mathcal{N}_{n-1}$ *obtained from* σ *by deleting* n *then* $I(\sigma) = n - j + I(\hat{\sigma})$.

To decompose a permutation, we need to partition its inversions into a number of disjoint sets. The following result enables us to organize this task.

Lemma 3.2.

(1) *There are* $[q^r]n!_q$ *permutations on* $\mathcal{N}_n$ *with* r *inversions.*

(2) *Let* $\Pi = (\pi_1, \ldots, \pi_k)$ *be an arbitrary ordered partition of type* $\underset{\sim}{i} = (i_1, \ldots, i_k)$. *There are* $[q^r](i_1!_q \cdots i_k!_q)$ *permutations with partition* Π *and* r *within-set inversions of type* $\underset{\sim}{i}$ *(an inversion* (t, s) *with* $t, s \in \pi_j$ *for some* $j = 1, \ldots, k$*).*

(3) *There are* $[q^r]\begin{bmatrix} n \\ \underset{\sim}{i} \end{bmatrix}_q$ *ordered partitions of* $\mathcal{N}_n$ *of type* $\underset{\sim}{i}$ *with* r *between-set inversions (an inversion* $(l, m) \in \pi_t \times \pi_s$ *with* $l > m$ *and* $t < s$*).*

Proof.

(1) Let $f_n(q)$ be the generating function for permutations on $\mathcal{N}_n$ with respect to inversions. By Proposition 3.1, we have $f_{n+1}(q) = (1 + q + \cdots + q^n) f_n(q)$ and the result follows.

(2) Immediate from Proposition 3.1(1) and (1).

(3) Construct all permutations on $\mathcal{N}_n$ as follows. Let Π be any ordered partition of type $\underset{\sim}{i}$. On each block of Π construct all permutations. Summing over all partitions Π of fixed type $\underset{\sim}{i}$, we have, from (2) and (1), $n!_q = i_1!_q \cdots i_k!_q G$, where G is the required generating function. The result follows.

For example, consider the permutation 45231 which has partition $(\{2, 4, 5\}, \{1, 3\})$ of type $(3, 2)$. There are 2 within-set inversions in 452 and 1 in 31 giving a total of 3 within-set inversions of type $(3, 2)$, namely $(4, 2)$, $(5, 2)$ and $(3, 1)$. There are 5 between-set inversions of type $(3, 2)$, namely $(2, 1)$, $(4, 1)$, $(4, 3)$, $(5, 1)$ and $(5, 3)$. The sum of the between-set and within-set inversions is 8, the number of inversions in the permutation 45231.

The q-binomial coefficient has properties analogous to those of the binomial coefficient, as the following two relationships indicate.

Consider the partitions of $\mathcal{N}_n$ of type $(k, n-k)$. By Lemma 3.2(3), the between-set generating function is $\binom{n}{k}_q$. This may be determined in another way. If n is in the k-set, then (n, j) is a between-set inversion for each j in the $(n-k)$-set. This case contributes $q^{n-k}\binom{n-1}{k-1}_q$ to the between-set generating function since the deletion of n leaves a partition of $\mathcal{N}_{n-1}$ of type $(k-1, n-k)$. If n is in the $(n-k)$-set, then n contributes no between-set inversions so the contribution in this case is $\binom{n-1}{k}_q$, since the deletion of n leaves a partition of type $(k, n-k-1)$. Thus

$$\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$$

As a second example, consider the partitions of $\mathcal{N}_{a+b}$ of type $(n, a+b-n)$. The between-set inversion generating function for the partitions is $\binom{a+b}{n}_q$, and this may be determined in another way. If (α_1, α_2) is a partition of $\{1, \ldots, a\}$ of type $(k, a-k)$ and (β_1, β_2) is a partition of $\{a+1, \ldots, a+b\}$ of type $(n-k, b+k-n)$, then $(\alpha_1 \cup \beta_1, \alpha_2 \cup \beta_2)$ is a partition of $\mathcal{N}_{a+b}$ of type $(n, a+b-n)$. Summing over all α_1, β_1 and k we obtain

$$\binom{a+b}{n}_q = \sum_{k=0}^{n} q^{(n-k)(a-k)}\binom{a}{k}_q\binom{b}{n-k}_q.$$

We note that these proofs are substantially the same as those for the corresponding binomial case $(q=1)$.

4. Cup- and Cap-permutations

The first result gives the generating function for cup- and cap-permutations with respect to number of inversions.

PROPOSITION 4.1. *There are $[z^k x^{n-k} q^j]Q_n(z, x)$ (respectively $[z^k x^{n-k} q^j]Q_n(x, z)$) cup- (respectively cap-) permutations on $\mathcal{N}_n$ of shape $(k, n-k)$ with j inversions, where*

(1) $Q_n(z, x) = (x+z)(x+qz)\cdots(x+q^{n-1}z)$

(2) $Q_n(z, x) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} z^k x^{n-k}$.

PROOF.

(1) We construct all cup-permutations on $\mathcal{N}_n$ by considering the insertion of n at either end of each cup-permutation on $\mathcal{N}_{n-1}$. Insertion of n on the left creates $n-1$ additional inversions [by Proposition 3.1(2)] and one extra element in the left-hand set (recorded by z). Insertion on the right gives no additional inversions and an extra element in the right-hand set (recorded by x). Thus

$$Q_n(z, x) = (x + zq^{n-1})Q_{n-1}(z, x) \quad \text{for } n \geqslant 2$$

and the result follows since $Q_1(z, x) = x + z$.

(2) We construct all cup-permutations of shape $(k, n-k)$ by considering all partitions of type $(k, n-k)$, with a decreasing permutation on the k-set and an increasing permutation on the $(n-k)$-set. Each of these decreasing permutations has $\binom{k}{2}$ inversions and each increasing permutation has no inversions. Thus, considering also the between-set inversions, summation over k yields

$$Q_n(z, x) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{k}{2}} z^k x^{n-k}.$$

For cap-permutations the argument is similar for (.2) and (.1) follows from the results for cup-permutations.

The next result is the main one.

THEOREM 4.2.

$$\sum_{k=0}^{n}\binom{n}{k}_q Q_k(x,y)Q_{n-k}(w,z)=\sum_{k=0}^{n}\binom{n}{k}_q Q_k(w,y)Q_{n-k}(x,z).$$

PROOF. Consider permutations of the form $(\alpha_1)_>(\alpha_2)_<(\alpha_3)_>(\alpha_4)_<$ where $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is any ordered partition of $\mathcal{N}_n$, in which the sizes of $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are recorded by x, y, w, z, respectively. We enumerate these permutations with respect to inversions in two ways.

First, note that $(\alpha_1)_>(\alpha_2)_<$ and $(\alpha_3)_>(\alpha_4)_<$ are cup-permutations on a partition of $\mathcal{N}_n$ of type $(k, n-k)$ for some $k\geqslant 0$. From Proposition 4.1, the generating functions for the cup-permutations with respect to within-set inversions are $Q_k(x,y)$ and $Q_{n-k}(w,z)$ for each partition of type $(k,n-k)$. Summing over all partitions of type $(k,n-k)$ we have the between-set inversions accounted for by $\binom{n}{k}_q$, from Lemma 3.2(3). Thus the generating function for these permutations with respect to inversions and the sizes of the sets $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ is given by the left-hand side of the statement of the theorem.

Second, we note that $(\alpha_1)_>$, $(\alpha_2)_<(\alpha_3)_>$ and $(\alpha_4)_<$ are decreasing, cap- and increasing permutations on a partition of $\mathcal{N}_n$ of type (i,k,l) for some $i,k,l\geqslant 0$, $i+k+l=n$. The decreasing permutation has $\binom{i}{2}$ inversions, the increasing permutation has no inversions and the cap-permutation has generating function $Q_k(w,y)$ for each partition of type (i,k,l). Summing over all partitions of type (i,k,l), we account for between-set inversions with $\begin{bmatrix} n \\ i,k,l\end{bmatrix}_q$, from Lemma 3.2(3). Thus the generating function for these permutations with respect to inversions and the sizes of $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ is

$$\sum_{i+k+l=n}\begin{bmatrix} n \\ i,k,l\end{bmatrix}_q q^{\binom{i}{2}}x^iz^lQ_k(w,y)=\sum_{k=0}^{n}\binom{n}{k}_q Q_k(w,y)\sum_{i=0}^{n-k}\binom{n-k}{i}_q q^{\binom{i}{2}}x^iz^{n-k-i}.$$

By Proposition 4.1(2), this is equal to the right-hand side of the statement of the theorem.

5. APPLICATIONS

The first result is the q-analogue of the binomial theorem (Goldman and Rota [5]).

COROLLARY 5.1. *Let*

$$P_n(x,z)=(x-z)(x-qz)\cdots(x-q^{n-1}z).$$

Then

$$P_n(x,z)=\sum_{k=0}^{n}\binom{n}{k}_q P_k(x,y)P_{n-k}(y,z).$$

PROOF. Direct from Theorem 4.2 with $w=-y$.

A number of identities for finite products follow immediately.

COROLLARY 5.2.

(1) $(x-1)(x-q)\cdots(x-q^{n-1})=\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}x^{n-k}$

(2) $x^n=\sum_{k=0}^{n}\binom{n}{k}_q(x-1)(x-q)\cdots(x-q^{k-1})$

(3) $(z-x)(z-qx)\cdots(z-q^{n-1}x)$

$$=\sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{k}{2}}\left\{\prod_{i=0}^{k-1}(x-q^i)\right\}\left\{\prod_{l=0}^{n-k-1}(z-q^{k+l})\right\}.$$

PROOF. In Corollary 5.1(1) set $y=0$, $z=1$; (2) set $y=1$, $z=0$; (3) replace x by xq^{n-1}, set $y=q^{n-1}$, and multiply through by $(-1)^n q^{-\binom{n}{2}}$.

To obtain the corresponding results when n tends to infinity, the following result is needed, which reveals the relationship to restricted integer partitions.

PROPOSITION 5.3.

$$\lim_{n\to\infty}\binom{n}{k}_q=(1-q)^{-1}(1-q^2)^{-1}\cdots(1-q^k)^{-1}.$$

PROOF. Consider a partition of $\mathcal{N}_n$ of type $(k,n-k)$ in which the k-set is $\{\alpha_1,\alpha_1+\alpha_2,\ldots,\alpha_1+\alpha_2+\cdots+\alpha_k\}$ where $\alpha_1,\ldots,\alpha_k\geqslant 1$ and $\alpha_1+\cdots+\alpha_k\leqslant n$. The number of between-set inversions due to the element $\alpha_1+\cdots+\alpha_i$ is $\alpha_1+\cdots+\alpha_i-i$. Thus the total number of between-set inversions for this partition is

$$\sum_{i=1}^{k}(\alpha_1+\cdots+\alpha_i-i)=\sum_{j=1}^{k}(k-j+1)(\alpha_j-1).$$

Thus, from Lemma 3.2(3) we have

$$\lim_{n\to\infty}\binom{n}{k}_q=\lim_{n\to\infty}\sum_{\substack{\alpha_1,\ldots,\alpha_k\geqslant 1\\ \alpha_1+\cdots+\alpha_k\leqslant n}}q^{(\alpha_k-1)+2(\alpha_{k-1}-1)+\cdots+k(\alpha_1-1)}$$
$$=(1-q)^{-1}(1-q^2)^{-1}\cdots(1-q^k)^{-1}.$$

This result may be used to obtain the following identities which involve infinite products.

COROLLARY 5.4.

(1) (Euler): $\prod_{i\geqslant 0}(1+tq^i)=1+\sum_{k\geqslant 1}\frac{q^{\binom{k}{2}}t^k}{(1-q)\cdots(1-q^k)}$

(2) (Euler): $\prod_{i\geqslant 0}(1-tq^i)^{-1}=1+\sum_{k\geqslant 1}\frac{t^k}{(1-q)\cdots(1-q^k)}$

(3) (Cauchy): $\prod_{i\geqslant 0}\frac{1-atq^i}{1-tq^i}=1+\sum_{k\geqslant 1}\frac{(1-a)(1-aq)\cdots(1-aq^{k-1})}{(1-q)(1-q^2)\cdots(1-q^k)}t^k$

(4) $\prod_{i\geqslant 0}(1-xq^i)=\sum_{k\geqslant 1}\frac{(-1)^k q^{\binom{k}{2}}}{(1-q)\cdots(1-q^k)}\left\{\prod_{i=0}^{k-1}(x-q^i)\right\}\left\{\left(\prod_{l\geqslant 0}(1-q^{k+l})\right)\right\}.$

PROOF. In Corollary 5.1, (1) set $x=1$, $y=0$, $z=-t$; (2) set $x=1$, $y=t$, $z=0$; (3) $x=1$, $y=t$, $z=at$, and let n tend to infinity using Proposition 5.3; (4) set $z=1$ in Corollary 5.2 (3), and let $n \to \infty$ using Proposition 5.3.

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